TENTAMEN IMAGE PROCESSING

22-04-2008

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A FORMULA SHEET IS INCLUDED ON PAGES 3-4

Put your name on all pages which you hand in, and number them. Write the total number of pages you hand in on the first page. Write clearly and not with pencil or red pen. You can answer in English or Dutch. ***Always motivate your answers.*** *You get 10 points for free. Success!*

Problem 1 (25 pt)
Consider a binary image X with 4-connected 1-pixels and 8-connected 0-pixels. Due to noise, the image contains isolated 1-pixels (1-pixels without 4-connected 1-pixels as neighbour), which we want to remove as much as possible. We consider three image operations to accomplish this task.

a. Using *thinning*. First explain the general idea of thinning and then give a structuring element pair $B = (B_1, B_2)$ which is suitable for the task.

b. Using *morphological reconstruction*. First explain the general idea of morphological reconstruction and then give a marker F and mask G which is suitable for the task.

c. Using a *median filter* with a 3×3 neighbourhood. First explain the general idea of median filtering and then explain why this is suitable for the task.

d. Discuss the relative merits of each of the three operations. Would you prefer one? Please motivate your answer.

Problem 2 (20 pt)
Two simple structures for representing images at several resolutions are image pyramids and wavelet transforms of images.

a. Explain the principle of image pyramids. Discuss the difference between an approximation pyramid and a prediction residual pyramid.

b. Explain the principle of discrete wavelet transforms for building a multiresolution representation of an image.

c. Given an $N \times N$ input image, an upper bound for the size of an image pyramid is given by $\frac{4}{3}N^2$, and for a discrete wavelet representation the size is N^2. Explain the difference between the size of the two representations.

c. How can wavelet transforms be used in image coding? What is the main advantage of wavelet coding compared to block coding based on the DCT (Discrete Cosine Transform)?

(continue on page 2)

Problem 3 (25 pt)
Highboost filtering of an input image $f(x, y)$ produces an output image $g(x, y)$ by

$$g(x, y) = f(x, y) + k \cdot g_{mask}(x, y), \quad (1)$$

where k is a constant with $k \geq 1$, and $g_{mask}(x, y)$ is a 'mask' image defined by

$$g_{mask}(x, y) = f(x, y) - \overline{f}(x, y)$$

with $\overline{f}(x, y)$ a smoothing of the input image $f(x, y)$.

a. What is the purpose of highboost filtering?

b. How does the result change if the constant k is increased? Argue why it is not profitable to take very large values of k.

c. Assume that the input image has size $M \times N$, and that the smoothing operation which produces $\overline{f}(x, y)$ is implemented by the following 3×3 mask:

$$\frac{1}{5} \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array}$$

In the frequency domain, equation (1) has the representation

$$G(u, v) = H(u, v)\, F(u, v),$$

where the transfer function $H(u, v)$ (centered around $(0, 0)$) has the form

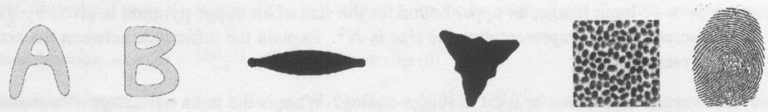
$$H(u, v) = k + 1 - \frac{k}{5}\Big[1 + 2\cos(2\pi u/M) + 2\cos(2\pi v/N)\Big]$$

Give a derivation of this formula.

d. The frequency domain transfer function satisfies $H(0, 0) = 1$. What property in the spatial domain does this formula correspond to?

e. What type of filter is highboost filtering: low-pass, high-pass, or other?

Problem 4 (20 pt)
Figure 1 shows a set of six images, both binary and grey value.



(a) (b) (c) (d) (e) (f)

Figure 1: *A diverse collection of images.*

We want to apply a set of descriptors to the objects in these images so that they can be distinguished on the basis of the numerical values taken by these descriptors.

Many solutions are possible, and there is no absolutely "correct" solution. Give a set of descriptors of your choice. Clearly indicate:

a. a short explanation of each descriptor in your set;

b. how your set of descriptors would allow to distinguish between the six objects.

Formula sheet

Co-occurrence matrix $g(i,j) = \{$no. of pixel pairs with grey levels (z_i, z_j) satisfying predicate $Q\}$, $1 \leq i, j \leq L$

Convolution, 2-D discrete $(f \star h)(x,y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, h(x-m, y-n)$, for $x = 0,1,2,\ldots,M-1, y = 0,1,2,\ldots,N-1$

Convolution Theorem, 2-D discrete $\mathcal{F}\{f \star h\}(u,v) = F(u,v)\, H(u,v)$

Distance measures Euclidean: $D_e(p,q) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2}$, City-block: $D_4(p,q) = |p_1 - q_1| + |p_2 - q_2|$, Chessboard: $D_8(p,q) = \max(|p_1-q_1|, |p_2-q_2|)$

Entropy, source $H = -\sum_{j=1}^{J} P(a_j) \log P(a_j)$

Entropy, estimated for L-level image: $\tilde{H} = -\sum_{k=0}^{L-1} p_r(r_k) \log_2 p_r(r_k)$

Error, root-mean square $e_{\rm rms} = \left[\frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\hat{f}(x,y) - f(x,y)\right)^2\right]^{\frac{1}{2}}$

Exponentials $e^{ix} = \cos x + i \sin x$; $\cos x = (e^{ix} + e^{-ix})/2$; $\sin x = (e^{ix} - e^{-ix})/2i$

Filter, inverse $\widehat{\mathbf{f}} = \mathbf{f} + \mathbf{H}^{-1}\mathbf{n}$, $\widehat{F}(u,v) = F(u,v) + \frac{N(u,v)}{H(u,v)}$

Filter, parametric Wiener $\widehat{\mathbf{f}} = \left(\mathbf{H}^t\mathbf{H} + K\,\mathbf{I}\right)^{-1} \mathbf{H}^t\mathbf{g}$, $\widehat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + K}\right] G(u,v)$

Fourier series of signal with period T: $f(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{i\frac{2\pi n}{T}t}$, with Fourier coefficients: $c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, e^{-i\frac{2\pi n}{T}t}\, dt$, $n = 0, \pm 1, \pm 2, \ldots$

Fourier transform 1-D (continuous) $F(\mu) = \int_{-\infty}^{\infty} f(t)\, e^{-i2\pi\mu t}\, dt$

Fourier transform 1-D, inverse (continuous) $f(t) = \int_{-\infty}^{\infty} F(\mu)\, e^{i2\pi\mu t}\, d\mu$

Fourier Transform, 2-D Discrete $F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-i2\pi(u\,x/M + v\,y/N)}$ for $u = 0,1,2,\ldots,M-1, v = 0,1,2,\ldots,N-1$

Fourier Transform, 2-D Inverse Discrete $f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v)\, e^{i2\pi(u\,x/M + v\,y/N)}$ for $x = 0,1,2,\ldots,M-1, y = 0,1,\ldots,N-1$

Fourier spectrum Fourier transform of $f(x,y)$: $F(u,v) = R(u,v) + i\, I(u,v)$, Fourier spectrum: $|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$, phase angle: $\phi(u,v) = \arctan\left(\frac{I(u,v)}{R(u,v)}\right)$

Gaussian function mean μ, variance σ^2: $G_\sigma(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$

Gradient $\nabla f(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$

Histogram $h(m) = \#\{(x,y) \in D : f(x,y) = m\}$. Cumulative histogram: $\mathrm{P}(\ell) = \sum_{m=0}^{\ell} h(m)$

Impulse, discrete $\delta(0) = 1, \delta(x) = 0$ for $x \in \mathbb{N}\backslash\{0\}$

Impulse, continuous $\delta(\infty) = 1, \delta(x) = 0$ for $x \neq 0$, with $\int_{-\infty}^{\infty} f(t)\, \delta(t - t_0)\, dt = f(t_0)$

Impulse train $s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$, with Fourier transform $S(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T})$

Laplacian $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

Laplacian-of-Gaussian $\nabla^2 G_\sigma(x,y) = -\frac{2}{\pi\sigma^4}\left(1 - \frac{r^2}{2\sigma^2}\right) e^{-r^2/2\sigma^2}$ $\quad (r^2 = x^2 + y^2)$

Morphology

Dilation $\delta_A(X) = X \oplus A = \bigcup_{a\in A} X_a = \bigcup_{x \in X} A_x = \{h \in E : \check{A}_h \cap X \neq \emptyset\}$,
where $X_h = \{x + h : x \in X\}$, $h \in E$ and $\check{A} = \{-a : a \in A\}$

Erosion $\varepsilon_A(X) = X \ominus A = \bigcap_{a\in A} X_{-a} = \{h \in E : A_h \subseteq X\}$

Opening $\gamma_A(X) = X \circ A := (X \ominus A) \oplus A = \delta_A\varepsilon_A(X)$

Closing $\phi_A(X) = X \bullet A := (X \oplus A) \ominus A = \varepsilon_A\delta_A(X)$

Hit-or-miss transform $X \circledast (B_1, B_2) = (X \ominus B_1) \cap (X^c \ominus B_2)$

Thinning $X \otimes B = X \backslash (X \circledast B)$, **Thickening** $X \odot B = X \cup (X \circledast B)$

Morphological reconstruction Marker F, mask G, structuring element B:
$X_0 = F$, $X_k = (X_{k-1} \oplus B) \cap G$, $\quad k = 1, 2, 3, \ldots$

Morphological skeleton Image X, structuring element B: $SK(X) = \bigcup_{n=0}^{N} S_n(X)$,
$S_n(X) = X \underset{n}{\ominus} B \setminus (X \underset{n}{\ominus} B) \circ B$, $S_0(X) = X$, with N the largest integer such that $S_N(X) \neq \emptyset$

Grey value dilation $(f \oplus b)(x, y) = \max\limits_{(s,t)\in B} [f(x - s, y - t) + b(s, t)]$

Grey value erosion $(f \ominus b)(x, y) = \min\limits_{(s,t)\in B} [f(x + s, y + t) - b(s, t)]$

Grey value opening $f \circ b = (f \ominus b) \oplus b$

Grey value closing $f \bullet b = (f \oplus b) \ominus b$

Morphological gradient $g = (f \oplus b) - (f \ominus b)$

Top-hat filter $T_{\text{hat}} = f - (f \circ b)$, **Bottom-hat filter** $B_{\text{hat}} = (f \bullet b) - f$

Sampling of continuous function $f(t)$: $\tilde{f}(t) = f(t)\, s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} f(t)\, \delta(t - n\Delta T)$.
Fourier transform of sampled function: $\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F(\mu - \frac{n}{\Delta T})$

Sampling theorem Signal $f(t)$, bandwidth $\mu_{\max}$: If $\frac{1}{\Delta T} \geq 2\mu_{\max}$, $f(t) = \sum_{n=-\infty}^{\infty} f(n\Delta T)\ \text{sinc}\left[\frac{t - n\Delta T}{n\Delta T}\right]$.

Sampling: downsampling by a factor of 2: $\downarrow_2 (a_0, a_1, a_2, \ldots, a_{2N-1}) = (a_0, a_2, a_4, \ldots, a_{2N-2})$

Sampling: upsampling by a factor of 2: $\uparrow_2 (a_0, a_1, a_2, \ldots, a_{N-1}) = (a_0, 0, a_1, 0, a_2, 0, \ldots, a_{N-1}, 0)$

Set, circularity ratio $R_c = \frac{4\pi A}{P^2}$ of set with area A, perimeter P

Set, diameter $\text{Diam}(B) = \max\limits_{i,j} [D(p_i, p_j)]$ with p_i, p_j on the boundary B and D a distance measure

Sinc function $\text{sinc}\,(x) = \frac{\sin(\pi x)}{\pi x}$ when $x \neq 0$, and $\text{sinc}\,(0) = 1$

Spatial moments of an $M \times N$ image $f(x, y)$: $m_{pq} = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} x^p\, y^q\, f(x, y)$, $\quad p, q = 0, 1, 2, \ldots$

Statistical moments of distribution $p(i)$: $\mu_n = \sum_{i=0}^{L-1} (i - m)^n\, p(i)$, $m = \sum_{i=0}^{L-1} i\, p(i)$

Signal-to-noise ratio, mean-square $\text{SNR}_{\text{rms}} = \frac{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \hat{f}(x,y)^2}{\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left(\hat{f}(x,y) - f(x,y)\right)^2}$

Wavelet decomposition with low pass filter h_ϕ, band pass filter h_ψ. For $j = 1, \ldots, J$:
Approximation: $c_j = \mathbf{H}c_{j-1} = \downarrow_2 (h_\phi * c_{j-1})$; Detail: $d_j = \mathbf{G}c_{j-1} = \downarrow_2 (h_\psi * c_{j-1})$

Wavelet reconstruction with low pass filter $\tilde{h}_\phi$, band pass filter $\tilde{h}_\psi$. For $j = J, J - 1, \ldots, 1$:
$c_{j-1} = \tilde{h}_\phi * (\uparrow_2 c_j) + \tilde{h}_\psi * (\uparrow_2 d_j)$

Wavelet, Haar basis $h_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $h_\psi = \frac{1}{\sqrt{2}}(1, -1)$, $\tilde{h}_\phi = \frac{1}{\sqrt{2}}(1, 1)$, $\tilde{h}_\psi = \frac{1}{\sqrt{2}}(1, -1)$